friction and divergence losses and accepting a frozen (or nearly frozen) flow. The optimum values of C_s obtained are shown in Fig. 3 along with the optimum value of nozzle wall angle as a function of velocity ratio. These values of C_s were obtained from studies similar to those that yielded the results shown in Fig. 2.

The optimum values of C_s shown in Fig. 3 were converted to a normalized net impulse and plotted as shown in Fig. 4 for a wall angle of 12. The lowest line shown represents the maximum computed impulse. Also a breakdown of the losses that produced the maximum impulse for each velocity ratio is indicated. The high level of these losses is apparent.

In summary, these studies suggest that, for conical nozzles, nonequilibrium losses are less important than wall friction for the flight conditions assumed. To investigate the effects of variations along a flight trajectory, further nozzle optimization studies of this type should be conducted for conical nozzles as well as other shapes. Because the results obtained depend on the skin friction utilized, experimental verification of nozzle skin-friction coefficients and means to reduce those coefficients should also be investigated.

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Implementation of an Optimal Adaptive Guidance Mode

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Introduction

THE guidance technique employed by multistaged and clustered space vehicles generally must accommodate a wide range of disturbances in the state vector of the vehicle during the accomplishment of its mission. Several of these guidance techniques depend upon the generation of trajectories optimized with respect to some specified criterion. Many methods exist for the generation of such trajectories in such a way that the possible disturbances are included. This leads to a tabulation of values of the optimal control laws for a specific mission. The tabulation of optimal control laws and their approximation by easily generated functional forms for use in actual control systems is referred to as a "flooding technique" by Kipniak,[2] and, in a more specialized form, as approximating a "statistical model" by Schmieder and Braud.[4] The digital computers with limited memories now being used for the guidance of space craft allow immediate realization of this form of optimal adaptive guidance (O.A.G.).

Presented as Preprint 63-417 at the AIAA Astrodynamics Conference, New Haven. Conn., August 19–21, 1963; revision received August 10, 1964. This work was sponsored, in part, under NASA Contract NAS8-2607.

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Method of Implementation

N sets of initial conditions are selected so as to encompass possible disturbances in the state of the vehicle during its mission and yet allow the successful completion of that mission in an optimal fashion. The optimized control laws are derived from the numerical solution of the N two-point boundary-value problems. The data for any one of the N trajectories are given in the form

$$u_i = [c_1(\mathbf{X}_i),\, c_2(\mathbf{X}_i), c_3(\mathbf{X}_i),\, \mathbf{X}_i]$$

where

$$\mathbf{X} = (X_1, X_2, \ldots, X_m)$$
$$\mathbf{X}_i = (X_{i1}, X_{i2}, \ldots, X_{im})$$

The subscript i refers to the ith time point on the trajectory and $c_K(\mathbf{X}_i)$ is the value of the Kth control function (K = 1, 2, 3) at the point $\mathbf{X}_i$. The X_v's ($v = 1, \ldots, m$) are the measurable state variables of the vehicle; e.g., position and velocity coordinates, time, and perhaps thrust divided by mass flow.

The present note reduces the problem of implementing O.A.G. to one of obtaining approximations of the c_K with linear or rational combinations of certain M basis functions, b_j, of the state variables

$$c_K \doteq \sum_{j=1}^{M} \alpha_{Kj} b_j \qquad (1)$$

or

$$c_K \doteq \Big(\sum_{j=1} \alpha_{Kj} b_j\Big)\Big(\sum_{j} \beta_{Kj} b_j\Big)^{-1} \qquad K = 1,2,3 \qquad (2)$$

The values of the basis functions b_j must be easily generated from values of the state variables $\mathbf{X}$. The coefficients α_{Kj} and β_{Kj} are fixed for each particular mission and stored in the memory of the onboard digital computer. Values of the control functions are calculated during the flight of the vehicle from the approximations (1) or (2). The interval between successive evaluations of the control functions depends on the mission and is limited by the response rate of the control hardware. Usually, for lunar and orbital missions, this interval will be well under 10 sec.

The special case where the basis functions b_j are monomials in the state variables has been considered in a number of reports.[4, 5, 7] This resulted in sufficiently accurate approximations to the control functions for practical use. A detailed error analysis of a particular mission using polynomial O.A.G. was given by Morgan.[3]

The effectiveness of this type of O.A.G. is directly related to the errors in the approximation of the c_K's. To determine the coefficients α_{Kj} and β_{Kj} of (1) and (2), a set of points u_i is selected for use in some numerical method of approximation. The magnitude and place of occurrence of the errors e_i are related to the u_i selected and to the form of the approximant (1) or (2). A third source of errors arises in the numerical method used to determine the unknown coefficients α_{Kj} and β_{Kj}.

When the forms of the approximants of the c_k's were polynomials, as in (1), a straightforward least-squares numerical method regularly failed to give accurate values of the α_{Kj} because of the associated system of ill-conditioned normal equations. Moreover, elimination of those basis functions b_j (monomials) that did not reduce the magnitude of the errors was difficult. These difficulties were resolved by the use of an orthonormal code.[1, 8] The code makes it possible to generate an orthonormal system of functions Q_j ($j = 1, \ldots, m$) from the set of basis functions b_j. The functions Q_j have the following properties:

1) The inner product (Q_j, Q_k) is 0 or 1 over the set of n points $\mathbf{X}_i$

$$(Q_j, Q_K) = \sum_{i=1}^{n} Q_j(\mathbf{X}_i)Q_K(\mathbf{X}_i) \tag{3}$$

$$= 0 \qquad (j \neq k) \tag{4}$$

$$= 1 \qquad (j = k) \tag{5}$$

2) A least-squares approximation of any function c_K over the set of n points $\mathbf{X}_i$ is given by

$$c_K \doteq \sum_{j=1}^{m} (Q_j, c_K)Q_j \tag{6}$$

In other words, if we let $b_j = Q_j$ in Eq. (1), then

$$\alpha_{Kj} = (Q_j, c_K) \tag{7}$$

Those Q_j's, whose inner products with the function being approximated are small, are eliminated. This follows from Bessel's inequality

$$\sum_{i=1}^{n} c_K^2(\mathbf{X}_i) \geq \sum_{j=1}^{s} (c_K, Q_j)^2 \qquad (s = 1, 2, \ldots) \tag{8}$$

This allows the number of α_{Kj}'s stored in the memory of the guidance computer to be minimized and the memory elements used for other essential activity.

3) If the region R over which the control functions c_K are defined is known in a geometrical sense, then the approximation may be improved by defining the inner product as the multiple integral over this region:

$$(b_j, b_K) = \int_R \cdots \int b_j(\mathbf{X})b_k(\mathbf{X})dX_1dX_2\ldots dX_m \tag{9}$$

Usually this region R can be approximated geometrically as a series of truncated hypercones and hypercylinders.[9, 10] The integral (9) may then be evaluated numerically from formulas developed by Stroud.[5] Additional information about the control functions such as values of partial derivatives may be incorporated into an approximation method[7] using an orthonormal code to yield highly accurate approximants.

Wheeler,[12] through an analysis of the Euler-Lagrange equations, expressed the tangent of an optimal thrust angle as a rational function of the instantaneous state variables as in Eq. (2). This model requires the evaluation of forty undetermined coefficients, a nonlinear problem in numerical analysis. The author suggested several years ago[8] that this type of problem can be handled by linear programming (L.P.). Moreover, L.P. allows one to specify the magnitudes of the errors of the approximant at predetermined points. Suzuki[6] developed a special L.P. method for nonlinear problems of approximation which overcomes some of the difficulties of ordinary L.P. systems written for commercial applications. In that paper, a solution to the problem of determining the greatest error is given, i.e., determine ϵ such that

$$\epsilon = \min_{\alpha_j, \beta_j} \max_{i} \left| \left[\sum^{M} \alpha_j b_j(\mathbf{X}_i)\right]\left[\sum^{M} \beta_j b_j(\mathbf{X}_i)\right]^{-1} - c_K(\mathbf{X}_i) \right| \tag{10}$$

This is of importance since a large value of ϵ would indicate a change of the form of the approximant should be made. Other approaches to error analysis are given in the paper by Weinberger and Golomb.[11]

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Magnetic Confinement of an Electric Arc in Transverse Supersonic Flow

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Introduction

A METHOD has been developed for the magnetic confinement of a d.c electric arc in an unheated supersonic air stream directed normal to the electric field. (Here a confined arc is one restrained within the limits of the freestream, at a fixed station.) The arc column, when confined by this method, exhibits remarkable spatial stability. The absence of appreciable fluctuations in the length and geometry of the positive column makes meaningful measurement of column voltage gradient, average electrical conductivity, and electrode fall voltage possible. The time available for

Presented as Preprint 64-26 at the AIAA Aerospace Sciences Meeting, New York, January 20–22, 1964; revision received August 13, 1964.

This research was sponsored by the Aerospace Research Laboratories, Office of Aerospace Research, U. S. Air Force, under Contract No. AF 33(657)-8819. The author wishes to express his sincere gratitude for the guidance and encouragement of A. M. Kuethe, the support and helpful suggestions of E. Soehngen and W. C. Roman, the valuable discussions with J. L. Amick, H. C. Early, A. F. Messiter, J. A. Nicholls, P. H. Rose, and R. A. Wolfe, and the laboratory assistance of C. Beneker, K. Richter, and J. Bennett.

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